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Time Evolution of an Infinite Number of Vortices in a Strip

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We study the time evolution of a system of vortices in a strip in the thermodynamic limit. We prove the existence and the uniqueness of the solution of the equation of motion for a regularized version of the usual vortex dynamics. We extend this result to a system of particles interacting in one dimension via a long-range potential.

KEY WORDS: Vortex theory; time evolution; thermodynamic limit.

1. INTRODUCTION

Vortex theory has been introduced in the 19th century to study some important features of fluid mechanics.(1) More recently it has been greatly developed for numerical purposes and rigorous connections between vortex theory and hydrodynamics in two dimensions have been established.(2,3) (Many papers have been devoted to the numerical analysis and it is not possible to quote all of them here.) Moreover such theory plays an important rôle in superfluidity and superconductivity (for an elementary introduction see Ref. 4).

The statistical mechanics of a system of vortices has been studied in connection with the theory of turbulence. The first paper in this direction is due to Onsager.(5) Recently Fröhlich and Ruelle have studied rigorously the thermodynamic limit for these systems.(6)

It is quite natural to study the time evolution of a vortex system. When the number of vortices involved is finite, many properties of the orbits have

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been obtained using both analytical and numerical methods.[3,7–11] In this paper we want to study the time evolution when the number of vortices is infinite, but the local density is finite. Such an evolution could not exist for two reasons: firstly, the infinite number of particles can produce some cooperative effect so that infinitely many vortices can arrive in a bounded region in a finite amount of time[12,13]; secondly, the singular character of the interaction can produce collapses involving a finite number of vortices. In this paper we overcome the first difficulty by restricting in a convenient way the initial condition (this is our main result). The second difficulty is not particular to the infinite case but it also arises in the few-body dynamics.[10,11] In our case it is avoided by introducing a regularized version of the interaction.

In this note we study a system of vortices moving in a strip. It is known that the time evolution in statistical mechanics depends in a crucial way on the dimension of the space[12] and the simplest models are one-dimensional ones, which is the case considered here. Furthermore a difficulty arises from the long range of the interaction between the vortices. Nevertheless in our case the presence of a boundary prevents strong interaction at long distances.

In Section 2 we introduce the model. In Section 3 we prove the existence and uniqueness of the time evolution problem, as well as some statistical estimates. Finally we state a result concerning a one-dimensional system of Newton particles with long-range interaction.

2. VORTEX MODEL

We consider the following Hamiltonian system: N particles, called vortices, whose positions are denoted by $q_i = (x_i, y_i) \in \Lambda \subset \mathbb{R}^2$, evolve according to the following equations:

$$\dot{q}_i(t) = \sum_{\substack{j=1 \\ j \neq i}}^{N} \nabla_i^{\perp} a_j g_\Lambda(q_i(t), q_j(t)) + \nabla_i^{\perp} a_i \tilde{\gamma}_\Lambda(q_i(t)) \tag{2.1}$$

where $a_i \in \mathbb{R}$ is the intensity of the ith vortex, $\nabla^{\perp} = [(\partial/\partial x_2), -(\partial/\partial x_1)]$, $g_\Lambda(q, q')$ is the Green function of the Laplace operator in Λ with Dirichlet boundary conditions (that is, $\Delta_q g_\Lambda(q, q') = -\delta(q - q')$, with $g_\Lambda(q, q') = 0$ for q or $q' \in \partial\Lambda$), and $\tilde{\gamma}_\Lambda(q) = \frac{1}{2}\gamma_\Lambda(q, q)$

$$\gamma_\Lambda(q, q') = g_\Lambda(q, q') + \frac{1}{2\pi} \ln(|q - q'|) \tag{2.2}$$

If Λ is not bounded we also require $g_\Lambda(q, q') \to 0$ as $|q| \to \infty$. The function $\tilde{\gamma}_\Lambda$ takes into account the interaction between a vortex and the boundary. When $\Lambda = \mathbb{R}^2$, $g_{\mathbb{R}^2} = (1/2\pi)\ln(|q - q'|)$ and $\tilde{\gamma}_\Lambda$ is absent.

The equation of motion (2.1) can be immediately put into a Hamiltonian form. If we define the energy of a vortex system by

$$H(\Lambda, N) = \frac{1}{2} \sum_{\substack{j=1 \\ i \neq j}}^{N} a_i a_j g_\Lambda(q_i, q_j) + \sum_{i=1}^{N} a_i^2 \tilde{\gamma}_\Lambda(q_i) \tag{2.3}$$

we have

$$\begin{aligned} a_i \dot{x}_i(t) &= \frac{\partial}{\partial y_i} H(\Lambda, N) \\ a_i \dot{y}_i(t) &= -\frac{\partial}{\partial x_i} H(\Lambda, N) \end{aligned} \tag{2.4}$$

Thus, for the evolution determined by (2.4) we have conservation of energy and Liouville's theorem holds.

This allows the construction of an equilibrium distribution and the use of methods of statistical mechanics.[6]

In this paper we want to study the dynamical problem (2.1) in the thermodynamic limit in which infinite vortices are considered and where the local density remains finite.

We study the problem in the strip $D = \mathbb{R} \times (0, a)$. The Green's function in D is

$$\begin{aligned} g_D(q, q') &= \frac{1}{\pi a} \sum_{n=1}^{\infty} \int_{-\infty}^{+\infty} dp \, \frac{\exp(-ip|x - x'|)}{p^2 + n^2} \sin \frac{n\pi}{a} y \sin \frac{n\pi}{a} y' \\ &= \frac{1}{a} \sum_{n=1}^{\infty} \frac{1}{n} \exp(-n|x - x'|) \sin \frac{n\pi}{a} y \sin \frac{n\pi}{a} y' \\ &= \frac{1}{4\pi} \ln \frac{\cosh(\pi/a)(x - x') - \cos(\pi/a)(y + y')}{\cosh(\pi/a)(x - x') - \cos(\pi/a)(y - y')} \end{aligned} \tag{2.5}$$

As we see, when $|q - q'| \to \infty$, $g_D \to 0$ exponentially; when $|q - q'| \to 0$, g_D diverges as a logarithm. This divergent character of the interaction can produce collapses. To avoid this difficulty we introduce a regularized version of the interaction which is finite together with its derivative and we study the following dynamical system (which is related but not equal to the original one):

$$\begin{aligned} \dot{q}_i(t) &= \sum_j a_j \nabla_i^\perp \bar{g}(q_i(t), q_j(t)) + a_i \nabla_i^\perp \bar{\gamma}(q_i(t)) \\ &= \sum_j a_j K_1(q_i(t), q_j(t)) + a_i K_2(q_i(t)) \end{aligned} \tag{2.6}$$

and we impose the following conditions:

(H.1) $\{a_j\}$ is bounded, i.e., $\sup_j a_j = A < \infty$.

(H.2) $K_1(q,q') \in C^1(D \times D)$ is bounded and there exists $\delta > 1$, such that $|K_1(q,q')| \leqslant \mathscr{F}_1(|q-q'|) = C/|q-q'|^\delta$ for some $C > 0$.

(H.3) There exists a nonincreasing, bounded, integrable function $\mathscr{F}_2(|q-q'|) \in C^1[0,+\infty)$ such that $|\nabla_q K_1(q,q')| \leqslant \mathscr{F}_2(|q-q'|)$ and $|\nabla_{q'} K_1(q,q')| \leqslant \mathscr{F}_2(|q-q'|)$.

(H.4) $K_2(q) \in C^1(D)$ and $|K_2(q)| \leqslant \bar{K}_2$, $|\nabla K_2(q)| \leqslant \bar{K}_3$, $\bar{K}_2, \bar{K}_3 > 0$.

Of course there are many ways to modify (2.5) to fulfill the above conditions. For instance, a possible choice for $\bar{g}$ and $\bar{\gamma}$ is ($\epsilon > 0$)

$$\begin{aligned} \bar{g}(q,q') &= \int_D \rho_\epsilon(|q''-q'|) g_D(q',q'')\,dq'' \\ \bar{\gamma}(q) &= \int_D \rho_\epsilon(q-q')\tilde{\gamma}_D(q')\,dq' \end{aligned} \tag{2.7}$$

where $\rho_\epsilon \in C^\infty(\mathbb{R}^1)$ is supported in $[-\epsilon,\epsilon]$; $\bar{g}$ describes the "interaction" between two bubbles of vorticity and $\bar{\gamma}$ the "interaction" between a bubble and the boundary. Another choice for the regularized interaction is

$$\bar{g}(q,q') = \frac{1}{\pi a}\sum_{n=1}^{M}\int_{-\infty}^{+\infty} dp\,\frac{\exp(-ip|x-x'|)}{p^2+n^2+\epsilon p^4}\sin\frac{n\pi}{a}y\sin\frac{n\pi}{a}y' \tag{2.8}$$

$\bar{\gamma}$ is bounded and satisfies (H.4).

This choice satisfies all the above conditions and, of course, when $M \to \infty$ and $\epsilon \to 0$, $\bar{g} \to g_D$.

3. INFINITE VOLUME DYNAMICS

We introduce the set Ω of locally finite labeled configuration. A configuration $\omega \in \Omega$ is a countable sequence $\omega = \{q_i, a_i\}$ of positions $q_i = (x_i, y_i) \in D$ and vorticity $a_i \leqslant A$ with a finite number of points in bounded regions. We suppose that the labeling is ordered in a way such that $i < j$ implies one of the following: $|x_i| < |x_j|$; $|x_i| = |x_j|$ and $x_i < x_j$; or $x_i = x_j$ and $y_i < y_j$.

For each $\omega = \{q_i, a_i\} \in \Omega$ we formally define the map $\omega \to \omega_t$ as a solution of the integral problem:

$$q_i(t) = q_i + \int_0^t F_i(s)\,ds \tag{3.1a}$$

where

$$F_i(s) = \sum_{j\in\mathbb{Z}} a_j K_1(q_i(s), q_j(s)) + a_i K_2(q_i(s)) \tag{3.1b}$$

and $K_1(q,q'), K_2(q)$ are the regularized forces satisfying conditions (H.2), (H.3), (H.4).

In order to give meaning to the initial value problem (3.1a, b) we need that the force on the ith vortex, i.e., the term $F_i(s)$ in (3.1a), remain bounded in bounded time intervals. It is known[12] that this might not be true for an arbitrarily chosen configuration, and it is necessary to make a restriction to a subset $\Omega_0 \subset \Omega$, which excludes the creation of singularities. On the other hand, the set Ω_0 must be large enough for thermodynamically interesting measures.

We set

$$\Omega_0 = \{\omega \in \Omega : Q(\omega) < \infty\} \tag{3.2}$$

with

$$Q(\omega) = \max\left\{1, \sup_{x} \sup_{d > \varphi(x)} \frac{N(\omega \mid R(x,d))}{d}\right\} \tag{3.3}$$

where $\varphi(x) = \max\{\ln|x|, 1\}$ and $N(\omega \mid R(x,d))$ is the number of vortices in the rectangle $R(x,d) = [x - d/2, x + d/2] \times (0,a)$.

As a consequence of (3.3) we have

$$N(\omega \mid R(x,d)) \leqslant Q(\omega)(\varphi(x) + d) \tag{3.4}$$

The set Ω_0 has full measure with respect to the Gibbs measure, as we will prove later.

We introduce a partial dynamics $\omega \to \omega_t^n$ useful to construct the solution of (3.1) (see Theorem 3.2):

$$\begin{aligned} q_i^n(t) &= q_i + \int_0^t F_i^n(s)\,ds \qquad &&\text{if} \quad |i| \leqslant n \\ q_i^n(t) &= q_i &&\text{if} \quad |i| > n \end{aligned} \tag{3.5a}$$

where

$$F_i^n(s) = \sum_{j \in \mathbb{Z}} a_j K_1(q_i^n(s), q_j^n(s)) + a_i K_2(q_i^n(s)) \tag{3.5b}$$

The map $\omega \to \omega_t^n = \{q_i^n(t), a_i\}$ describes the evolution of the first n vortices in the field generated by themselves, by the others which remain fixed, and by the boundary.

We first prove the following theorem that gives a bound for the displacement and the forces on the vortices, independent of the partial dynamics ω^n.

Theorem 3.1. Let $\omega = \{q_i, a_i\}_{i \in \mathbb{Z}} \in \Omega_0$ and suppose that the interactions $K_1(q, q'), K_2(q)$ in (3.5a, b) satisfies the conditions (H.2), (H.4). Then there exist two positive, continuous, increasing functions $h_1(\omega, |t|), h_2(\omega, |t|)$, independent of n, such that, for each $t \in \mathbb{R}$, we have

$$|q_i^n(s) - q_i| \leqslant h_1(\omega, |t|)\varphi(q_i) \tag{3.6a}$$

$$|F_i^n(s)| \leqslant h_2(\omega, |t|)\varphi(q_i), \qquad |s| \leqslant |t| \tag{3.6b}$$

Proof. We have, for each $t > 0$ (we omit the suffix n),

$$|q_i(t) - q_i| \leqslant \int_0^t |F_i(s)|\, ds$$

In order to obtain a bound for $F_i(s)$ we construct a sequence of rectangular shells, centered in the geometrical point $x_i(s)$. Hence we put $b_0 = 0$, $b_n = \exp n$ (for each $n \geqslant 1$), and we define, for $n \geqslant 1$

$$A_n = \{x \in \mathbb{R} : b_{n-1} \leqslant |x_i(s) - x| \leqslant b_n\}, \qquad I_n = A_n \times (0,a) \quad (3.7a)$$

$$B_n = \{x \in \mathbb{R} : 0 \leqslant |x_i(s) - x| \leqslant b_n\}, \qquad R_n = B_n \times (0,a) \quad (3.7b)$$

Next we define for $t \geqslant 0$

$$s_i(t) = \sup_{s \in [0,t]} |q_i(s) - q_i| \tag{3.8a}$$

$$d(t) = \sup_{i \in \mathbb{Z}} s_i(t)/\varphi(q_i) \tag{3.8b}$$

We consider a vortex $q_j(t)$ that lies in the rectangle R_k at time t, and estimate in which rectangle $\tilde{R}_k \supset R_k$ this vortex can lie at time 0. We note that a vortex that at time t is in the geometrical point $q(t) = (x(t), y(t))$ was at time t at most in the rectangle

$$\left[x(t) - d(t)\varphi(\tilde{x}(t)), x(t) + d(t)\varphi(\tilde{x}(t))\right] \times (0,a)$$

where $\tilde{x}(t)$ is the solution of the equation:

$$\tilde{x}(t) = |x(t)| + d(t)\varphi(\tilde{x}(t)) \tag{3.9}$$

Then, using the fact that the solution $\tilde{x}(t)$ is a monotone nondecreasing function of $|x(t)|$ we have that, if $x_i(t) \in B_k$, then:

$$x_i(0) \in \left[x_i(t) - b_k - d(t)\varphi(\tilde{z}_k(t)), x_i(t) + b_k + d(t)\varphi(\tilde{z}_k(t))\right]$$

where $\tilde{z}_k(t)$ is the solution of

$$\tilde{z}_k(t) = |x_i(t)| + b_k + d(t)\varphi(\tilde{z}_k(t)) \tag{3.10}$$

Then

$$\tilde{R}_k = \left[x_i(t) - b_k - d(t)\varphi(\tilde{z}_k(t)), x_i(t) + b_k + d(t)\varphi(z_k(t))\right] \times (0,a) \tag{3.11}$$

With this estimate we obtain a rough bound on the number of vortices in the rectangular shell I_k:

$$\begin{aligned} N(\omega_t \mid I_k) &\leqslant N(\omega_t \mid R_k) \leqslant N(\omega \mid \tilde{R}_k) \\ &\leqslant Q(\omega)\left[\varphi(x_i(t)) + \exp(k) + d(t)\varphi(\tilde{Z}_k(t))\right] \\ &\leqslant Q(\omega)\left\{\varphi(x_i(t)) + \exp(k) + d(t)c_1\left[\varphi(d(t)) + \varphi(x_i(t) + b_k)\right]\right\} \\ &\leqslant c_2 Q(\omega)\left[\varphi(x_i(t)) + \exp(k) + d(t)\left[\varphi(d(t)) + \varphi(x_i(t)) + k\right]\right\} \end{aligned} \tag{3.12}$$

In (3.12) we have used (3,4), the bound:

$$\varphi(x(t)) \leqslant c_1\left[\varphi(d(t)) + \varphi(x(t))\right] \tag{3.13}$$

and $\varphi(x+y) \leqslant \varphi(x) + \varphi(y)$.

Now, from (3.12) we obtain a bound on the forces. First we note that the following series are convergent [see Eq. (H.2)]:

$$\begin{aligned} S_1 &= \mathscr{F}_1(0) + \sum_{k=1}^{\infty} \mathscr{F}_1(e^k) \\ S_2 &= \sum_{k=1}^{\infty} k\mathscr{F}_1(e^k) \\ S_3 &= \mathscr{F}_1(0) + \sum_{k=1}^{\infty} e^k\mathscr{F}_1(e^k) \end{aligned} \tag{3.14}$$

Hence using (3.5b) and (3.12),

$$\begin{aligned} |F_i(s)| &\leqslant \sum_{j\in\mathbb{Z}} |a_j||K(q_i(s), q_j(s))| + |a_i||K_2(q_i(s))| \\ &\leqslant A \sum_{k=1}^{\infty} \sup_{q\in I_k} |K(q, q_i(s))| N(\omega_s \mid R_k) + A\overline{K}_2 \\ &\leqslant Ac_3\, Q(\omega) \sum_{k=1}^{\infty} \mathscr{F}_1(b_{k-1})\{\varphi(x_i(s)) + \exp(k) \\ &\qquad\qquad + d(s)\left[\varphi(d(s)) + \varphi(x_i(s)) + k\right]\} \\ &\leqslant Ac_4\, Q(\omega)\{S_1\left[\varphi(x_i(s)) + \varphi(d(s))d(s) + d(s)\varphi(x_i(s))\right] \\ &\qquad + S_2 d(s) + S_3\} \end{aligned} \tag{3.15}$$

We insert this bound in

$$d(t) \leqslant \int_0^t \sup_{i\in\mathbb{Z}} \frac{|F_i(s)|}{\varphi(q_i)}\, ds \tag{3.16}$$

and use

$$\frac{\varphi(x_i(s))}{\varphi(x_i)} \leqslant c\varphi(d(s)) \tag{3.17}$$

to obtain

$$\begin{aligned} d(t) &\leqslant Ac_5\, Q(\omega) \int_0^t ds\, \{S_1\left[\varphi(d(s)) + d(s)\varphi(d(s))\right] + S_2 d(s) + S_3\} \\ &\leqslant A \text{ const } Q(\omega) \int_0^t ds \left[\varphi(d(s)) + 1\right]\left[d(s) + 1\right] \end{aligned} \tag{3.18}$$

The differential equation associated with the integral inequality (3.18) has a global solution and hence, using Gronwall's lemma, we prove (3.6a). Next (3.6b) is obtained by inserting the bound (3.6a) in inequality (3.15). ∎

With the *a priori* bound of Theorem 3.1 we can prove the existence and uniqueness of the solution of the dynamical problem (2.6).

Theorem 3.2. Let $\omega = \{q_i, a_i\}_{i \in \mathbb{Z}} \in \Omega_0$. There exists a sequence of integers n_k such that

$$\lim_{k \to \infty} q_i^{n_k}(t) = q_i(t) \tag{3.19}$$

where $q_i^n(t)$ are defined in (3.5), and

$$q_i(t) = q_i + \int_0^t F_i(s)\, ds \tag{3.20}$$

where

$$F_i(s) = \sum_{j \in \mathbb{Z}} a_j K_1(q_i(s), q_j(s)) + a_i K_2(q_i(t)) \tag{3.21}$$

Hence there exists a solution $\omega_t = \{q_i(t), a_i\}$.

Further the solution $q_i(t)$ satisfies the bound:

$$|q_i(t) - q_i| \leqslant h(\omega, |t|)\varphi(q_i) \tag{3.22}$$

where $h(\omega, |t|)$ is a positive continuous increasing function of t. Finally, this solution is unique in the set of functions having the bound (3.22).

Proof. The existence of the solution and the property (3.19), (3.22) are consequences of the bound (3.6a, b) and of the Ascoli–Arzelà theorem.

To prove the uniqueness we use an iterative procedure. Let $\omega^1(t) = \{q_i^1(t), a_i\}$ and $\omega^2(t) = \{q_i^2(t), a_i\}$ be two solutions of (3.20) which satisfy the bound (3.22) starting from the same initial condition $\omega = \{q_i, a_i\}_{i \in \mathbb{Z}} \in \Omega_0$. We want to estimate the difference

$$q_i^1(t) - q_i^2(t) = \int_0^t ds \left[F_i(\omega^1(s)) - F_i(\omega^2(s)) \right] \tag{3.23}$$

For this purpose we separately treat the contribution to the force from the vortices near q_i and far from q_i. The first term is controlled using (H.3), the second one is estimated by using (H.2).

We introduce the sequence r_k defined as

$$r_k = r_0(1 + \alpha)^k \tag{3.24}$$

where $\alpha > 0$, and $r_0 > 0$ (r_0 will be chosen later on).

We suppose now $|x_i| \leqslant r_k$ and we write

$$F_i(\omega^1(s)) - F_i(\omega^2(s)) = \sum_{|x_j| \leqslant r_{k+1}} a_j\left[K_1(q_i^1(s), q_j^1(s)) - K_1(q_i^2(s), q_j^2(s))\right] + \sum_{|x_j| > r_{k+1}} a_j\left[K_1(q_i^1(s), q_j^1(s)) - K_1(q_i^2(s), q_j^2(s))\right] + a_i\left[K_2(q_i^1(s)) - K_2(q_i^2(s))\right] \tag{3.25}$$

Using Lemma 1 (see below), we control the first sum on the right-hand side of (3.25):

$$\sum_{|x_j| \leqslant r_{k+1}} a_j\left[K_1(q_i^1(s), q_j^1(s)) - K_1(q_i^2(s), q_j^2(s))\right] + a_i\left[K_2(q_i^1(s)) - K_2(q_i^2(s))\right] \leqslant A(s)\varphi(r_{k+1})\delta(r_{k+1}, s) \tag{3.26}$$

where

$$\delta(r_k, t) = \sup_{s \in [0,t]} \sup_{|x_j| \leqslant r_k} |q_j^1(s) - q_j^2(s)| \tag{3.27}$$

We want to estimate the second term in the sum (3.25). We choose r_0 such that

$$r_0 \geqslant 4\alpha^{-1} d(t)\varphi(r_0(1+\alpha)) \tag{3.28}$$

Then if $|x_i| \leqslant r_k$ and $|x_j| \geqslant r_{k+1}$ we have that

$$|x_i^{(\frac{1}{2})}(s) - x_j^{(\frac{1}{2})}(s)| \geqslant \alpha r_k/2 \qquad \text{for each} \quad s \in [0, t] \tag{3.29}$$

Hence

$$\sum_{|x_j| \geqslant r_{k+1}} a_j |K_1(q_i^1(s), q_j^2(s)) - K_1(q_i^2(s), q_j^2(s))| \leqslant \sum_{|x_i(s) - x_j(s)| > \alpha r_k/2} a_j|(\cdots)| \leqslant B(s)(2/(\alpha r_k))^{\delta - 1}\varphi(r_k) \tag{3.30}$$

where $B(s)$ is a positive, continuous, nonincreasing function of s. To obtain this last equation we must use the bound (3.22) and (H.2) (see Lemma 2).

Combining the estimates (3.26) and (3.30) we have

$$|F_i(\omega^1(s)) - F_i(\omega^2(s))| \leqslant A(t)\varphi(r_{k+1})\delta(r_{k+1}) + B(t)\left(\frac{2}{\alpha r_k}\right)^{\delta - 1}\varphi(r_k) \tag{3.31}$$

Inserting (3.31) into the integral Eq. (3.23) we obtain

$$\delta(r_k, t) \leqslant A(t)\varphi(r_{k+1})\int_0^t ds\, \delta(r_{k+1}, s)\, ds + tB(t)\left(\frac{2}{\alpha r_k}\right)^{\delta-1} \varphi(r_k) \quad (3.32)$$

Now we iterate (3.32):

$$\begin{aligned}
\delta(r_0, t) &\leqslant A^n(t) \prod_{k=1}^{n} \varphi(r_k) \int_0^t ds_2 \int_0^{s_1} ds_l \ldots \int_0^{s_{n-1}} ds_n\, \delta(r_n, s_n) \\
&\quad + tB(t) \sum_{l=1}^{n} [A(t)]^l \left(\prod_{k=1}^{l} \varphi(r_k)\right)\left(\frac{2}{\alpha r_k}\right)^{\delta-1} \\
&\quad \times \varphi(r_l) \int_0^t ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{l-1}} ds_l + tB(t)\left(\frac{2}{\alpha r_0}\right)^{\delta-1} \varphi(r_0) \\
&\leqslant A^n(t)\varphi^n(r_n)\frac{t^n}{n!} + tB(t)\left[\left(\frac{2}{\alpha r_0}\right)^{\delta-1} \varphi(r_0)\right. \\
&\qquad \left. + \sum_{l=1}^{n} A^l(t)\varphi^{l+1}(r_l)\left(\frac{2}{\alpha r_l}\right)^{\delta-1} \frac{t^l}{l!}\right] \\
&\leqslant A^n(t)\left[(2\varphi(r_0))^n + (2n\varphi(1+\alpha))^n\right]\frac{t^n}{n!} \\
&\quad + tB(t)\left(\frac{2}{\alpha r_0}\right)^{\delta-1}\left\{\varphi(r_0) + \sum_{l=1}^{\infty} A^l(t)\left[(2\varphi(r_0))^{l+1}\right.\right. \\
&\qquad \left. + (2(l+1)\varphi(1+\alpha))^{l+1}\right] \\
&\qquad \left. \times (1+\alpha)^{-l(\delta-1)} \frac{t^l}{l!}\right\}
\end{aligned} \quad (3.33)$$

We can evaluate this expression when $n \to \infty$. The first term on the right-hand side goes to zero when $t \leqslant 1/6A(t)\varphi(1+\alpha)$ and the second is independent of n. Now we let $r_0 \to \infty$ and the last term vanishes when t is small enough; more precisely

$$t \leqslant \min\left[\frac{(\delta-1)(1+\alpha)^{\delta-1}}{2A(t)}, \frac{(1+\alpha)^{\delta-1}}{6A(t)\varphi(1+\alpha)}\right]$$

In conclusion we have proved that for small t and for each r

$$\delta(r, t) = \sup_{s \in [0,t]} \sup_{|x_j| \leqslant r} |q_j^1(s) - q_j^2(s)| = 0$$

i.e., we have uniqueness for small time. Because of the uniformity in t of the *a priori* bound (3.22) we can prove the result for all time. ■

Lemma 1. Let K_1 satisfy (H.3), let $|x_i| \leqslant r_k$, and let $q_i(s)$ satisfy (3.20) and the bound (3.22); then

$$\sum_{|x_j| \leqslant r_{k+1}} |K_1(q_i^1(s), q_j^1(s)) - K_1(q_i^2(s), q_j^2(s)) \leqslant A(s)\varphi(r_{k+1})\delta(r_{k+1}, s) \tag{3.34}$$

where $A(s)$ is a bounded positive increasing function of s.

Proof.

$$\begin{aligned}
&\sum_{|x_j| \leqslant r_{k+1}} |K_1(q_i^1(s), q_j^1(s)) - K_1(q_i^1(s), q_j^2(s))| \\
&\quad \leqslant \sum_{|x_j| \leqslant r_{k+1}} |K_1(q_i^1(s), q_j^1(s)) - K_1(q_i^1(s), q_j^2(s))| \\
&\qquad + |K_1(q_i^1(s), q_j^2(s)) - K_1(q_i^2(s), q_j^2(s))|
\end{aligned} \tag{3.35}$$

We study the first term on the right-hand side of (3.35). Let A_n, I_n, B_n, R_n be the sequences of shells and rectangles centered in the point $q_i(s)$ defined in (3.7a, b), with the choice

$$b_0 = 0, \qquad b_n = \Delta \exp(n) \qquad \text{for} \quad n \geqslant 1 \quad \text{and} \quad \Delta = 2d(s)\varphi(r_{k+1})$$

As a consequence of the bound (3.22), if $q_j^1(s) \in I_n$, then $q_j^2(s) \in I_{n-1} \cup I_n \cup I_{n+1}$, and also if $q_j^1(s) \in I_n$, then $q_j^2(s) \in R_{n+1}$.

Hence

$$\begin{aligned}
&\sum_{|x_j| \leqslant r_{k+1}} |K_1(q_i^1(s), q_j^1(s)) - K_1(q_i^1(s), q_j^2(s))| \\
&\quad \leqslant \sum_{n=1}^{\infty} N(\omega^1(s) \mid I_n) \\
&\quad \times \sup_{\substack{q_j^1(s) \in I_n \\ q_j^2(s) \in I_{n-1} \cup I_n \cup I_{n+1}}} |K_1(q_i^1(s), q_j^1(s)) - K_1(q_i^1(s), q_j^2(s))|
\end{aligned} \tag{3.36}$$

By use of the mean value theorem we have

$$\begin{aligned}
&\sup_{\substack{q_j^1(s) \in I_n \\ q_j^2(s) \in I_{n-1} \cup I_n \cup I_{n+1}}} |K_1(q_i^1(s), q_j^1(s)) - K_1(q_i^1(s), q_j^2(s))| \\
&\quad \leqslant \sup_{\substack{q_j^1(s) \in I_n \\ q_j^2(s) \in I_{n-1} \cup I_n \cup I_{n+1}}} |q_i^1(s) - q_j^2(s)| |\nabla_q K_1(q_i(s), q)|_{q \in I_{n-1} \cup I_n \cup I_{n+1}} \\
&\quad \leqslant \delta(r_{k+1}, s) \mathscr{F}_2(b_{n-2})
\end{aligned} \tag{3.37}$$

We have also

$$N\big(\omega^1(s), I_n\big) \leqslant N\big(\omega^1(s), R_n\big) \leqslant Q\big(\omega^1(s)\big)\big[\varphi(q_i(s)) + \Delta e^n\big]$$
$$\leqslant Q\big(\omega^1(s)\big)\big[\varphi(r_k) + d(n)\varphi(r_k) + 2d(s)\varphi(r_{k+1})\exp(n)\big]$$
$$\leqslant a_1(s)\varphi(r_{k+1})\exp(n) \tag{3.38}$$

Inserting (3.38) and (3.37) into (3.36) we have

$$\sum_{|q_j| \leqslant r_{k+1}} |K_1\big(q_i^1(s), q_j^1(s)\big) - K_1\big(q_i^1(s), q_j^2(s)\big)|$$
$$\leqslant a_2(s)\delta(r_{k+1}, s)\varphi(r_{k+1})\left[\sum_{n=2}^{\infty} \exp(n)\mathscr{F}_2(b_{n-2}) + \mathscr{F}_2(s)\right] \tag{3.39}$$

Using the nonincreasing behavior of $\mathscr{F}_2$ we have

$$\sum_{n=2}^{\infty} \exp(n)\mathscr{F}_2(b_{n-2}) \leqslant \sum_{n=2}^{\infty} \exp(n)\mathscr{F}_2(d(s)\exp(n-2))$$

This sum is convergent and so in (3.35) the first term satisfies the statement. The second term in (3.35) can be treated in a similar way.

Lemma 2. Let K_1 satisfy (H.2) and $q_i(t)$ satisfy (3.20) and the bound (3.22). Then we have for $s \in [0, t]$

$$\sum_{\substack{|x_i(s) - x_j(s)| \geqslant r \\ |x_i| \leqslant r_k}} |K_1(q_i(s), q_j(s))| \leqslant B(t)\varphi(r_k)\left(\frac{1}{r}\right)^{\delta - 1} \tag{3.40}$$

for a positive, bounded constant $B(t)$.

Proof. Let I_n, R_n the sequence (3.7a, b) with $b_n = r\exp(n)$. We have

$$\sum_{\substack{|x_i(s) - x_j(s)| \geqslant r \\ |x_i| \leqslant r_k}} |K_1(q_i(s), q_j(s))| \leqslant \sum_{n=1}^{\infty} N(\omega(s) \mid I_n)\left(\frac{1}{b_{n-1}}\right)^{\delta}$$
$$\leqslant \left(\frac{1}{r}\right)^{\delta} \sum_{n=1}^{\infty} N(\omega(s) \mid R_n)\left[\frac{1}{\exp(n-1)}\right]^{\delta}$$
$$\leqslant c_1 Q(\omega(s))\left(\frac{1}{r}\right)^{\delta} \sum_{n=1}^{\infty} \left[\varphi(r_k) + d(t)\varphi(r_k) + r\exp(n)\right]\left[\frac{1}{\exp(n-1)}\right]^{\delta}$$
$$\leqslant B(s)\varphi(r_k)\left(\frac{1}{r}\right)^{\delta - 1} \quad \blacksquare$$

So we have proved the existence and the uniqueness of the dynamics when the initial conditions belong to Ω_0.

This result is relevant in statistical mechanics if Ω_0 is large enough. For this purpose we introduce an additional hypothesis on the interaction:

(H.5) $\bar{g}(q, q')$ is the integral kernel of a positive quadratic operator on $L^2(D, \text{Lebesgue})$; $\bar{g}$ and $\bar{\gamma}$ are bounded.

An example of $\bar{g}$ satisfying (H.5) is given in (2.8).

So we shall prove that

$$\mu(\Omega_0) = \mu(\{\omega \in \Omega \mid Q(\omega) < \infty\}) = 1 \tag{3.41}$$

where μ is the Gibbs measure obtained as the local weak limit of finite volume Gibbs measures μ_Λ, for some sequence of compact regions Λ invading the strip D. We define the finite volume Gibbs measure for a system of vortices in the domain $\Lambda \subset D$, with the vorticity distributed according to some finite, positive measure $d\lambda(a)$ with the property $d\lambda(a) = d\lambda(-a)$:

$$\mu_\Lambda(d\omega) = Z_\Lambda^{-1}(\beta, z) \sum_{n=0}^{\infty} \frac{z^n}{n!} \prod_{i=1}^{n} d\lambda(a_i)$$

$$\times \exp\left[-\frac{\beta}{2} \sum_{\substack{i,j=1\\ i\neq j}}^{n} a_i a_j \bar{g}(q_i, q_j) - \beta \sum_{i=1}^{n} a_i^2 \bar{\gamma}(q_i)\right] dq_1 \ldots dq_n \tag{3.42}$$

It can be proved that the correlation function

$$\rho_\Lambda^n(\beta, Z) = Z_\Lambda^{-1}(\beta, z) \sum_{p=0}^{\infty} \frac{z^{n+p}}{p!} \int_{\mathbb{R}^p} \prod_{i=1}^{p} d\lambda(a_i)$$

$$\times \int_{\Lambda^p} \exp\left[-\frac{\beta}{2} \sum_{\substack{i,j=1\\ i\neq j}}^{n} a_i a_j \bar{g}(q_i, q_j)\right] dq_{n+1} \ldots dq_{n+p} \tag{3.43}$$

has the bound

$$\rho_\Lambda^n(z, \beta) \leqslant \eta^n \qquad \text{with} \quad \eta > 0. \tag{3.44}$$

We give here only a short sketch of the proof following Refs. 6 and 13. For simplicity we suppose $a_i = \pm 1$ and $\beta = 1$.

We introduce a Gaussian process $\phi(q)$ with mean 0 and covariance

$\bar{g}(q,q')$. In terms of this process the correlation function can be written as

$$\rho_\Lambda^n(q_1, \ldots, q_n, a_1, \ldots, a_n; z)$$
$$= \frac{1}{Z_\Lambda(\beta, z)} \sum_{p=0}^{\infty} \frac{z^{n+p}}{p!\,2^p} \sum_{\substack{a_j = \pm 1 \\ j=n+1,\ldots,n+p}} \int_{\Lambda^p} dq_{n+1} \ldots dq_{n+p}$$
$$\times \exp\left[-\frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n+p} a_i a_j \bar{g}(q_i, q_j) - \sum_{j=1}^{n+p} ai^2 \bar{\gamma}(q_i) \right]$$
$$= z^n \exp\left(\sum_{j=1}^{n} c(q_i) \right) \left\langle \exp\left[z \int e^{c(q)} \cos\phi(q)\, dq \right] \exp\left[i \sum_{i=1}^{n} a_i \phi(q_i) \right] \right\rangle$$
$$\times \left[\left\langle \exp\left[z \int e^{c(q)} \cos\phi(q)\, dq \right] \right\rangle \right]^{-1} \tag{3.45}$$

where

$$c(q) = \tfrac{1}{2}\, \bar{g}(q,q) - \bar{\gamma}(q) \tag{3.46}$$

and $\langle \cdot \rangle$ denotes the expectation with respect to the Gaussian process (Sine–Gordon transformation).

From (3.45) and (3.46) we obtain the bound (3.44) with

$$\eta^n = z^n \exp\left(\sum_{i=1}^{n} \tfrac{1}{2}\, \bar{g}(q_i, q_i) - \bar{\gamma}(q_i) \right) \tag{3.47}$$

Let $N_\Delta(\omega)$ denote the number of vortices in the domain Δ, and $p_\Delta(n)$ its probability distribution:

$$p_\Delta(n) = \mu_\Lambda(\{\omega \mid N_\Delta(\omega) = n\}) \tag{3.48}$$

From the bound (3.44) we also have [$V(\Delta)$ = volume of Δ]

$$p_\Delta(n) \leqslant \frac{\eta^n}{n!} V^n(\Delta) \tag{3.49}$$

Using (3.49) we obtain the bound

$$\mu_\Lambda(\exp[N_\Delta(\omega)]) = \sum_{n=0}^{\infty} \exp(n) p_\Delta(n) \leqslant \exp[3\eta V(\Delta)] \tag{3.50}$$

and for the limit measure

$$\mu(\exp[N(\omega \mid R(x,d))]) \leqslant \text{const} \exp(3\eta d a) \tag{3.51}$$

Now, by the Tchebychev inequality we have for any $m \in \mathbb{N}$

$$\mu(\{\omega \mid N(\omega \mid R(x,d)) \geqslant md\}) \leqslant \text{const} \exp[d(3a\eta - m)] \tag{3.52}$$

Hence

$$\mu(\{\omega \mid Q(\omega) \geqslant m) \leqslant \mu\left(\bigcup_{k\in\mathbb{N}} \bigcup_{\substack{n>\varphi(k)\\ n\in\mathbb{N}}} \{\omega \mid N(\omega \mid R(k,n)) \geqslant c_1 mn\}\right)$$
$$\leqslant C_2 \sum_{k=1}^{\infty} \sum_{n>\varphi(k)}^{\infty} \exp\left[n(3a\eta - c_1 m)\right] \tag{3.53}$$

where c_1, c_2 are suitable constants.

From (3.49) it follows that

$$\sum_{n=1}^{\infty} \mu(\{\omega \mid Q(\omega) \geqslant m\}) < \infty \tag{3.54}$$

and hence we have the assertion $\mu(\Omega_0) = 1$ by using the Borel–Cantelli lemma.

Finally we note that we can use the same technique of Theorems 3.1 and 3.2 for proving the existence and uniqueness of solutions of the Newton equation of motion of an infinite one-dimensional system of particles interacting via long-range forces.

Theorem 3.3. Let Ω be the set of locally, finite configurations $\omega = \{q_i, p_i\}_{i\in\mathbb{Z}}$ with $q_i, p_i \in \mathbb{R}$. Newton equations of motion take the form

$$q_i(t) = q_i + \frac{p_i}{m} t + \int_0^t (t-s) F_i(s)\, ds \tag{3.55}$$

where

$$F_i(s) = -\sum_{j\neq i} \nabla_i \phi(|q_i - q_j|) \tag{3.56}$$

and ϕ satisfies the following hypotheses:

(1) $\phi'(r)$ is bounded and there exists $\delta > 1$ such that $|\phi'(r)| \leqslant \mathrm{const}/r^{\delta}$, for $r > 0$.

(2) There exists a nonincreasing, bounded, integrable function $\mathscr{F}_2(r) \in C^1[0, +\infty)$ such that $|\phi''(r)| \leqslant \mathscr{F}_2(r)$ for $r > 0$. Let $\Omega_0 = \{\omega \in \Omega \mid Q(\omega) < \infty\}$, where

$$\begin{aligned}
Q(\omega) &= \max\{ Q_1(\omega), Q_2(\omega)\}\\
Q_1(\omega) &= \max\left\{ \sup_{x\in\mathbb{R}} \sup_{d>q(x)} \frac{N(\omega \mid R(x,d))}{d}, 1\right\}\\
Q_2(\omega) &= \max\left\{ \sup_{i\in\mathbb{Z}} |p_i(\omega)|/\varphi(q_i(\omega)), 1\right\}
\end{aligned} \tag{3.57}$$

Then, if $\omega \in \Omega_0$, there exists a unique solution of (3.55).

If we introduce the additional hypothesis:

(3) Superstability: $\phi(r) = \phi_1(r) + \phi_2(r)$, where $\phi_2(r) \geqslant 0$, $\varphi_2(0) > 0$ and continuous in 0, $\varphi_1(r)$ is stable, i.e., there exists a $B \geqslant 0$ such that

$$\sum_{\substack{i,j=1 \\ i<j}}^{n} \phi_2(|q_i - q_j|) \geqslant -nB$$

for each n and $q \in \mathbb{R}$;
we can prove that $\mu(\Omega_0) = 1$ where μ is the Gibbs measure.

We remark that this result can be used to study a two-dimensional Coulomb system with a continuous charge density moving in a strip with Dirichlet boundary conditions.

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